COHOMOLOGICAL CHARACTERIZATION OF HYPERQUADRICS OF ODD DIMENSIONS IN CHARACTERISTIC TWO

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ABSTRACT. We consider characterizations of projective varieties in terms of their tangents. S. Mori established the characterization of projective spaces in arbitrary characteristic by ampleness of tangent bundles. J. Wahl characterized projective spaces in characteristic zero by cohomological condition of tangent bundles; in addition, he remarked that a counter-example in characteristic two is constructed from odd-dimensional hyperquadrics Q_{2n-1} with $n > 1$. This is caused by existence of a common point in $\mathbb{P}^{2n}$ which every embedded tangent space to the quadric contains. In general, a projective variety in $\mathbb{P}^N$ is said to be strange if its embedded tangent spaces admit such a common point in $\mathbb{P}^N$. A non-linear smooth projective curve is strange if and only if it is a conic in characteristic two (E. Lluis, P. Samuel). S. Kleiman and R. Piene showed that a non-linear smooth hypersurface in $\mathbb{P}^N$ is strange if and only if it is a quadric of odd-dimension in characteristic two. In this paper, we investigate complete intersection varieties, and prove that, a non-linear smooth complete intersection variety in $\mathbb{P}^N$ is strange if and only if it is a quadric in $\mathbb{P}^N$ of odd dimension in characteristic two; these conditions are also equivalent to non-vanishing of 0-cohomology of (-1)-twist of the tangent bundle.

1. INTRODUCTION

S. Mori [7] established the characterization of projective spaces in characteristic $p \geqslant 0$ by ampleness of tangent bundles. The work has motivated approaches via tangential properties to characterizing several projective varieties. J. Wahl [10] characterized projective spaces in $p = 0$ by cohomological condition of tangent bundles; in addition, he remarked that a counter-example in $p = 2$ is constructed from odd-dimensional hyperquadrics Q_{2n-1} with $n > 1$ [10, p. 316]. This is caused by existence of a common point $v \in \mathbb{P}^{2n}$ which every embedded tangent space to the quadric contains (Remark 2.7). In general, a projective variety $X \subset \mathbb{P}^N$ is said to be *strange* if its embedded tangent spaces admit such a common point $v \in \mathbb{P}^N$ (see §2, for more details).

It is classically known that a non-linear smooth projective curve $X \subset \mathbb{P}^N$ is strange if and only if X is a conic in $\mathbb{P}^N$ in $p = 2$ (E. Lluis [5], P. Samuel [9]). In higher-dimensions, S. Kleiman and R. Piene [4, Theorem 7] focused on

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hypersurfaces, and showed that a non-linear smooth hypersurface $X \subset \mathbb{P}^N$ is strange if and only if X is a quadric in $\mathbb{P}^N$ of odd dimension in $p = 2$.

In this paper, we investigate whether a smooth complete intersection variety (without quadrics) can be strange, and answer it negatively by examining a parameter space of strange complete intersection varieties. In consequence, we have:

Theorem 1.1. *Let X be a smooth projective variety which is embedded in $\mathbb{P}^N$ as a non-linear complete intersection, and denote by $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_X$ the very ample line bundle. Then the followings are equivalent:*

(a) *X is strange in $\mathbb{P}^N$,*
(b) *X is a quadric in $\mathbb{P}^N$ of odd dimension in $p = 2$.*

Moreover, in the case where $\dim(X) \geqslant 2$*, the conditions* (a) *and* (b) *are equivalent to*

(c) $H^0(X, T_X(-1)) \neq 0$.

This paper is organized as follows: In §2.1, studying relation between strangeness and cohomology, we show the equivalence "(a) $\Leftrightarrow$ (c)" (for the exceptional case $\dim(X) = 1$, see Remark 2.10). In §2.2, we analyze defining polynomials of strange varieties. We say that a projective variety in $\mathbb{P}^N$ is an $(e^1, \dots, e^r)$-complete intersection if it is scheme-theoretically equal to an intersection of r hypersurfaces of degrees $e^1, \dots, e^r$, where $e^1, \dots, e^r$ are r positive integers. In (2) of §3, we define an irreducible parameter space $\mathcal{H}^v$ of $(e^1, \dots, e^r)$-complete intersection varieties being strange for v. In order to show the implication "(a) $\Rightarrow$ (b)", it is essential to consider the case where $e^k > 1$ for any k and where $\mathcal{H}^v$ is *not* equal to the parameter space of quadrics of odd dimensions in $p = 2$. In §3.1, we will construct an $(e^1, \dots, e^r)$-complete intersection variety $X_0 \subset \mathbb{P}^N$ having an isolated singular point $\alpha \neq v$. In §3.2, we will take the incidence variety $I \subset \mathcal{H}^v \times \mathbb{P}^N$ parametrizing pairs of strange complete intersection varieties and their singular points, and in addition, take an irreducible component $\Lambda \subset I$ whose subset parametrizes the orbit of the pair (X_0, α) under automorphisms of $\mathbb{P}^N$ with fixed point v. Calculating the dimension of Λ, we will show that the projection $\Lambda \to \mathcal{H}^v$ is surjective in the case. This means that every $(e^1, \dots, e^r)$-complete intersection variety belonging to $\mathcal{H}^v$ is singular (Theorem 3.8), yielding Theorem 1.1.

2. Preliminary

Let $X \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field K of characteristic $p \geqslant 0$. We say that X is *strange for a point* $v \in \mathbb{P}^N$ if $v \in \mathbb{T}_x X$ for any smooth point $x \in X$, where $\mathbb{T}_x X \subset \mathbb{P}^N$ is the embedded tangent space to X at x. We simply say that X is *strange* in $\mathbb{P}^N$ if X is strange for some point of $\mathbb{P}^N$.

Let $(z_0 : z_1 : \cdots : z_N)$ be the homogeneous coordinates on $\mathbb{P}^N$. We denote by $f_{z_j} := \partial f / \partial z_j$ for a homogeneous polynomial $f \in K[z_0, z_1, \dots, z_N]$.

Proposition 2.1. *Let $X \subset \mathbb{P}^N$ be a hypersurface defined by a homogeneous polynomial f, and let $v = (1 : 0 : \cdots : 0) \in \mathbb{P}^N$. Then X is strange for v if and only if f_{z_0} is the zero polynomial.*

Proof. Let $a \in X$ be a smooth point, where recall that $\mathbb{T}_a X = (f_{z_0}(a) z_0 + \cdots + f_{z_N}(a) z_N = 0) \subset \mathbb{P}^N$. Then $v \in \mathbb{T}_a X$ if and only if $f_{z_0}(a) = 0$. Hence X is strange for v if and only if $f_{z_0}|_X = 0$. Here, the latter condition $f_{z_0}|_X = 0$ means that f_{z_0} is contained in the ideal $(f) \subset K[z_0, \dots, z_N]$, and then we have $f_{z_0} = 0$ because of $\deg(f_{z_0}) < \deg(f)$. □

Note that, in the case where $X \subset \mathbb{P}^N$ is a degenerate subvariety and is contained in an m-dimensional linear subvariety L of $\mathbb{P}^N$, X is strange in $\mathbb{P}^N$ if and only if X is strange in $L \simeq \mathbb{P}^m$.

Example 2.2. Let $X \subset \mathbb{P}^N$ be a smooth quadric, i.e., a smooth projective variety of degree 2 in $\mathbb{P}^N$. Then X is strange if and only if $\dim(X)$ is odd and $p = 2$. The reason is as follows: It is sufficient to consider the case where X is non-degenerate; thus we set $N = \dim(X) + 1$. Let f be the defining equation of X. Choosing suitable coordinates $(z_0 : z_1 : \cdots : z_N)$ on $\mathbb{P}^N$, we can assume that

$$f = \begin{cases} z_0^2 + z_1 z_2 + z_3 z_4 + \cdots + z_{N-1} z_N & \text{if } \dim(X) \text{ is odd,} \\ z_0 z_1 + z_2 z_3 + \cdots + z_{N-1} z_N & \text{if } \dim(X) \text{ is even.} \end{cases}$$

(a) Assume that $\dim(X)$ is odd and $p = 2$. Then $f_{z_0} = 2z_0 = 0$; hence it follows from Proposition 2.1 that X is strange for $v = (1 : 0 : \cdots : 0)$.

(b) Assume that $\dim(X)$ is even or $p \neq 2$. Let us consider the Gauss map $\gamma : X \to (\mathbb{P}^N)^\vee$ sending $a \mapsto \mathbb{T}_a X$, where $(\mathbb{P}^N)^\vee = \mathbb{G}(N-1, \mathbb{P}^N)$ is the space of hyperplanes of $\mathbb{P}^N$. Indeed, we can describe γ by

$$a \mapsto (f_{z_0}(a) : \cdots : f_{z_N}(a)).$$

Then, by assumption, γ is isomorphic with $\gamma^*(\mathcal{O}(1)) = \mathcal{O}(1)$, and $\gamma(X)$ is also a smooth quadric hypersurface in $(\mathbb{P}^N)^\vee$. We denote by $u^* \subset (\mathbb{P}^N)^\vee$ the set of hyperplanes containing a point $u \in \mathbb{P}^N$. Then u^* is a hyperplane of $(\mathbb{P}^N)^\vee$. If X is strange for some $u \in \mathbb{P}^N$, then we have $\gamma(X) \subset u^*$, a contradiction. Hence X is not strange.

We say that a projective variety $X \subset \mathbb{P}^N$ is a *cone* with vertex $v \in \mathbb{P}^N$ if the line $\overline{xv}$ is contained in X for any $x \in X$. If X is a cone with vertex v, then X is strange for v.

Remark 2.3. In $p = 0$, if X is strange for v, then X is a cone with vertex v. The reason is as follows: Let $\pi_v : \mathbb{P}^N \setminus \{ v \} \to \mathbb{P}^{N-1}$ be the linear projection from v, and let $d_x \pi_v : T_x \mathbb{P}^N \to T_{\pi_v(x)} \mathbb{P}^{N-1}$ be the linear map between Zariski tangent spaces. For any smooth point $x \in X$ with $x \neq v$, since $T_x \overline{xv} \subset T_x X \cap \ker(d_x \pi_v)$, we have $\dim(d_x \pi_v(T_x X)) = \dim(X) - 1$. It follows from $p = 0$ that $\dim(\pi_v(X \setminus \{ v \})) = \dim(X) - 1$; hence X is a cone with vertex v.

Remark 2.4. Let $X \subset \mathbb{P}^N$ be a projective variety being strange for a point v. Then we immediately have the following properties:

(a) If X is smooth and $L \subset \mathbb{P}^N$ is a hyperplane not containing v, then $X \cap L$ is smooth. This is because, for each point $x \in X \cap L$, it follows from $v \in \mathbb{T}_x X$ that $\mathbb{T}_x X \not\subset L$; hence $X \cap L$ is smooth at x and $\mathbb{T}_x(X \cap L) = \mathbb{T}_x(X) \cap L$.

(b) Let $\pi_z : \mathbb{P}^N \setminus \{ z \} \to \mathbb{P}^{N-1}$ be a linear projection from a point $z \in \mathbb{P}^N$ with $z \neq v$. Then the image $Y \subset \mathbb{P}^{N-1}$ of X under π_z is strange for $\pi_z(v)$. This is because, $\mathbb{T}_{\pi_z(x)} Y$ contains $\pi_z(\mathbb{T}_x X)$ for a general point $x \in X$.

2.1. **Section of 0-cohomology of (-1)-twist of a tangent bundle.** Let $X \subset \mathbb{P}^N$ be a smooth quasi-projective variety, and let $\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_X$. We identify $\mathbb{P}^N$ with $(H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \setminus 0)/(K \setminus 0)$, the projectivization of the dual vector space of $H^0(\mathbb{P}^N, \mathcal{O}(1))$. Let

$$\hat{v} \subset H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee$$

be the one-dimensional vector subspace corresponding to $v \in \mathbb{P}^N$. Considering the Euler sequence $0 \to \mathcal{O}_{\mathbb{P}^N} \to H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\xi} T_{\mathbb{P}^N} \to 0$, we can define a composite homomorphism s_v of bundles on $\mathbb{P}^N$ by

$$s_v : \hat{v} \otimes \mathcal{O}_{\mathbb{P}^N}(1) \hookrightarrow H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O}_{\mathbb{P}^N}(1) \xrightarrow{\xi} T_{\mathbb{P}^N}.$$

Proposition 2.5. *Let $X \subset \mathbb{P}^N$ be a smooth quasi-projective variety. Then X is strange for v if and only if $s_v|_X$ factors through $T_X \subset T_{\mathbb{P}^N}|_X$. Hence, in this case, $s_v|_X$ gives a nonzero section of $H^0(X, T_X(-1)) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(1), T_X)$.*

To show this, we consider $\mathcal{P}^1_X = \mathcal{P}^1_X(\mathcal{O}_X(1))$, the bundle of principal parts of $\mathcal{O}_X(1)$ of first order, which gives an exact sequence

$$0 \to \Omega^1_X \to \mathcal{P}^1_X \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0.$$

Taking the dual of this, we have the following sequence with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & {\mathcal{P}^1_X}^\vee \otimes \mathcal{O}_X(1) \arrow[r] \arrow[d] & T_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee \otimes \mathcal{O}_X(1) \arrow[r, "\xi|_X"] & T_{\mathbb{P}^N}|_X \arrow[r] & 0
\end{tikzcd} \tag{1}$$

Considering the middle column arrow, since the projectivization of ${\mathcal{P}^1_X}^\vee \otimes k(x) \subset H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee$ corresponds to $\mathbb{T}_x X \subset \mathbb{P}^N$, we find that $\hat{v} \subset {\mathcal{P}^1_X}^\vee \otimes k(x)$ if and only if $v \in \mathbb{T}_x X$. In particular, the following holds:

Lemma 2.6. *The subbundle $\hat{v} \otimes \mathcal{O}_X \subset H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_X$ is contained in ${\mathcal{P}^1_X}^\vee$ if and only if X is strange for v.*

Proof of Proposition 2.5. If X is strange for v, then it follows from Lemma 2.6 that $s_v|_X$ is equal to the composite map, $\hat{v} \otimes \mathcal{O}_X(1) \hookrightarrow {\mathcal{P}^1_X}^\vee \otimes \mathcal{O}_X(1) \to T_X$. If X is not strange for v, then $\hat{v} \not\subset {\mathcal{P}^1_X}^\vee \otimes k(x)$ for some x, and then the image of $s_v(x) : \hat{v} \to T_x \mathbb{P}^N$ is not contained in $T_x X$. □

Remark 2.7. If a smooth projective variety X is strange and is not isomorphic to a projective space, then it follows from Proposition 2.5 that X gives a counter-example in $p > 0$ of the statement of Wahl's cohomological characterization of projective spaces. For example, smooth quadrics in $p = 2$ whose dimensions are odd and $\geqslant 3$ (Example 2.2). (One dimensional quadrics, i.e., conics, are still isomorphic to $\mathbb{P}^1$.)

In addition, we can restate strangeness of tangents as a cohomological condition, as follows:

Corollary 2.8. *Let X be a smooth projective variety, and let $\iota : X \hookrightarrow \mathbb{P}^N$ be an embedding with a very ample line bundle $\mathcal{O}_X(1) := \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$. Denote by $\mathcal{P}^1_X = \mathcal{P}^1_X(\mathcal{O}_X(1))$ the bundle of principal parts of $\mathcal{O}_X(1)$ of first order. Then X is strange in $\mathbb{P}^N$ if and only if $H^0(X, {\mathcal{P}^1_X}^\vee) \neq 0$.*

Proof. The "only if" part follows immediately from Lemma 2.6. To show the "if" part, we consider the case where there exists a nonzero section $s \in H^0(X, {\mathcal{P}^1_X}^\vee)$. By the middle column of (1), s is regarded as a nonzero section of $H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee$, which gives a point $v \in \mathbb{P}^N$ such that $K \cdot s = \hat{v}$. Then X is strange for v. □

Now, let us consider the case where X is a complete intersection.

Proposition 2.9. *Let X be a smooth projective variety embedded in $\mathbb{P}^N$ as a complete intersection. Assume that $\dim(X) \geqslant 2$. Then X is strange in $\mathbb{P}^N$ if and only if $H^0(X, T_X(-1)) \neq 0$.*

Remark 2.10. The case where $\dim(X) = 1$ is exceptional: Assume that X is a smooth $(e^1, \dots, e^{N-1})$-complete intersection curve in $\mathbb{P}^N$. Then $T_X = \mathcal{O}_X(N + 1 - \sum_{1 \leqslant k \leqslant N-1} e^k)$, which implies that $H^0(X, T_X(-1)) \neq 0$ if and only if X is a conic or line (in any $p \geqslant 0$). We recall that a smooth conic in $p \neq 2$ is not strange.

Lemma 2.11. *Let $X = X^1 \cap \cdots \cap X^r \subset \mathbb{P}^N$ be an $(N-r)$-dimensional complete intersection variety with hypersurfaces $X^1, \dots, X^r$ of degrees $e^1, \dots, e^r$. Then $H^j(X, \mathcal{O}_X(i)) = 0$ for $0 < j < N - r$ and all $i \in \mathbb{Z}$.*

Proof. Since $H^j(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(i)) = 0$ for $0 < j < N$ and all i, it follows from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^N}(-e^1) \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{X^1} \to 0$ that $H^j(X^1, \mathcal{O}_{X^1}(i)) = 0$ for $0 < j < N - 1$ and all i. Similarly, it follows from the exact sequence $0 \to \mathcal{O}_{X^1}(-e^2) \to \mathcal{O}_{X^1} \to \mathcal{O}_{X^1 \cap X^2} \to 0$ that $H^j(X^1 \cap X^2, \mathcal{O}_{X^1 \cap X^2}(i)) = 0$ for $0 < j < N - 2$ and all i. Inductively, we have $H^j(X, \mathcal{O}_X(-i)) = 0$ for $0 < j < N - r$ and all i. □

Proof of Proposition 2.9. The "only if" part was shown in Proposition 2.5. Let us prove the "if" part under the hypothesis. From Lemma 2.11 and $N - r = \dim(X) \geqslant 2$, we have $H^1(X, \mathcal{O}_X(-1)) = 0$. By the first row of (1), it follows that

$$H^0(X, {\mathcal{P}^1_X}^\vee) \to H^0(X, T_X(-1))$$

is surjective. Hence the assertion follows from Corollary 2.8. □

2.2. **Defining polynomials of a strange complete intersection variety.** Let $e^1, \dots, e^r$ be r integers greater than 1. We recall that $X \subset \mathbb{P}^N$ is an $(e^1, \dots, e^r)$-complete intersection if X is scheme-theoretically equal to an intersection of r hypersurfaces of degrees $e^1, \dots, e^r$, i.e., the defining homogeneous ideal $I_X \subset K[z_0, z_1, \dots, z_N]$ of X is generated by r homogeneous polynomials of degrees $e^1, \dots, e^r$. We generalize Proposition 2.1, as follows:

Proposition 2.12. *Let $X \subset \mathbb{P}^N$ be an $(e^1, \dots, e^r)$-complete intersection variety which is strange for a point $v = (1 : 0 : \cdots : 0)$. Then I_X is generated by r homogeneous polynomials $f^1, \dots, f^r$ of degrees $e^1, \dots, e^r$ such that $f^k_{z_0}$ is the zero polynomial for $1 \leqslant k \leqslant r$.*

To prove the above statement, we first show the following lemma:

Lemma 2.13. *Let $X \subset \mathbb{P}^N$ be a projective variety which is strange for $v = (1 : 0 : \cdots : 0)$. For a homogeneous polynomial $g \in I_X$ of degree e, there exists a homogeneous polynomial $\tilde{g} \in I_X$ of degree e, such that $\tilde{g}_{z_0} = 0$ and that two ideals $(g, z_0), (\tilde{g}, z_0) \subset K[z_0, z_1, \dots, z_N]$ coincide.*

Proof. For a homogeneous polynomial $g \in I_X$, we have $g_{z_0} \in I_X$. The reason is as follows: There is nothing to prove if g is the zero polynomial. Let g be nonzero, and denote by $Y := (g = 0) \subset \mathbb{P}^N$, the hypersurface defined by g. For any $a \in X$, it follows that $v \in \mathbb{T}_a X \subset \mathbb{T}_a Y$; then, since $v = (1 : 0 : \cdots : 0)$ and $\mathbb{T}_a Y = (g_{z_0}(a) z_0 + \cdots + g_{z_N}(a) z_N = 0)$, we have $g_{z_0}(a) = 0$. Hence $g_{z_0} \in I_X$.

Applying the above argument inductively, we have $\partial^j g / \partial z_0^j \in I_X$ for any $j > 0$. Now let

$$\tilde{g} := g + \sum_{j=1}^{p-1} \frac{(-1)^j z_0^j}{j!} \cdot \frac{\partial^j g}{\partial z_0^j},$$

which is contained in I_X and satisfies that $(g, z_0) = (\tilde{g}, z_0)$. In addition, we have $\tilde{g}_{z_0} = 0$, because of

$$\begin{aligned} \frac{\partial}{\partial z_0}\left(\sum_{j=1}^{p-1} \frac{(-1)^j z_0^j}{j!} \cdot \frac{\partial^j g}{\partial z_0^j}\right) &= \sum_{j=1}^{p-1} \frac{(-1)^j z_0^{j-1}}{(j-1)!} \cdot \frac{\partial^j g}{\partial z_0^j} + \sum_{j=1}^{p-1} \frac{(-1)^j z_0^j}{j!} \cdot \frac{\partial^{j+1} g}{\partial z_0^{j+1}} \\ &= -\frac{\partial g}{\partial z_0} + \frac{(-1)^{p-1} z_0^{p-1}}{(p-1)!} \cdot \frac{\partial^p g}{\partial z_0^p} = -\frac{\partial g}{\partial z_0}, \end{aligned}$$

where $\partial^p g / \partial z_0^p = 0$ since p is the characteristic of the ground field. □

Remark 2.14. The operation making $\tilde{g}$ with $\tilde{g}_{z_0} = 0$ is naturally appeared in an algorithm of derivation kernel computation (see [2, p. 27]; for positive characteristic, see [8]).

Proof of Proposition 2.12. Let I_X be generated by homogeneous polynomials $f^1, \dots, f^r$ of degrees $e^1, \dots, e^r$. From Lemma 2.13, for each $1 \leqslant k \leqslant r$, we have a homogeneous polynomial $\tilde{f}^k$ of degree e^k satisfying that $\tilde{f}^k \in I_X$, $\tilde{f}^k_{z_0} = 0$, and $(f^k, z_0) = (\tilde{f}^k, z_0)$. We set $\tilde{X} \subset \mathbb{P}^N$ to be the complete intersection defined

by $\tilde{f}^1, \dots, \tilde{f}^r$. Then we have $X \subset \tilde{X}$ and $X \cap (z_0 = 0) = \tilde{X} \cap (z_0 = 0)$ for the hyperplane $(z_0 = 0) \subset \mathbb{P}^N$.

Let $x \in X \cap (z_0 = 0)$. Then $x \notin (z_i = 0)$ for some i with $1 \leqslant i \leqslant N$. We assume $i = 1$, and set $\tilde{F}^k := \tilde{f}^k / z_1^{e^k}$, $Z_0 := z_0/z_1$, which are functions on the affine open subset $(z_1 \neq 0) \simeq \mathbb{A}^N$. Then we have

$$\mathcal{O}_{\tilde{X},x}/(Z_0) = \mathcal{O}_{\mathbb{P}^N,x}/(\tilde{F}^1, \dots, \tilde{F}^r, Z_0),$$

where the left hand side is equal to $\mathcal{O}_{X,x}/(Z_0)$. Since $X \cap (z_0 = 0)$ is smooth at x as in Remark 2.4(a), the local ring $\mathcal{O}_{X,x}/(Z_0)$ is regular, and so is $\mathcal{O}_{\tilde{X},x}/(Z_0)$. It follows from [6, Theorem. 14.2] that $\tilde{F}^1, \dots, \tilde{F}^r, Z_0$ give a subset of a regular system of parameters of $\mathcal{O}_{\mathbb{P}^N,x}$. In particular, $\mathcal{O}_{\tilde{X},x} = \mathcal{O}_{\mathbb{P}^N,x}/(\tilde{F}^1, \dots, \tilde{F}^r)$ is regular. This implies that $\tilde{X}$ is smooth and of dimension $N - r$ around x. It follows that X is an irreducible component of $\tilde{X}$. Since $\deg(X) = \deg(\tilde{X})$, we have $X = \tilde{X}$. □

Corollary 2.15. *Let $X \subset \mathbb{P}^N$ be an $(e^1, \dots, e^r)$-complete intersection variety. Assume that X is strange for a point v and assume that $e^k < p$ for any k with $1 \leqslant k \leqslant r$. Then X is a cone with vertex v.*

Proof. Changing coordinate, we can assume that $v = (1 : 0 : \cdots : 0) \in \mathbb{P}^N$. Then it follows from Proposition 2.12 that X is defined by homogeneous polynomials $f^1, \dots, f^r$ of degrees $e^1, \dots, e^r$ such that $f^k_{z_0} = 0$ for $1 \leqslant k \leqslant r$. If $f^k \notin K[z_1, \dots, z_N]$, then the inequality $e^k < p$ implies $f^k_{z_0} \neq 0$, a contradiction. Hence $f^k \in K[z_1, \dots, z_N]$ for any k, which means that X is a cone with vertex v. □

3. Parameter space of strange complete intersection varieties

Let $\mathcal{H}_e = |\mathcal{O}_{\mathbb{P}^N}(e)|$ be the projectivization of $H^0(\mathbb{P}^N, \mathcal{O}(e))$, which parametrizes hypersurfaces of $\mathbb{P}^N$ of degree e. For a point $v \in \mathbb{P}^N$, we denote by $\mathcal{H}^v_e$ the subset of $\mathcal{H}_e$ which parametrizes hypersurfaces being strange for v. Changing homogeneous coordinates $(z_0 : z_1 : \cdots : z_N)$ on $\mathbb{P}^N$, we can assume

$$v = (1 : 0 : \cdots : 0).$$

Then we have $\mathcal{H}^v_e = \{ f \in \mathcal{H}_e \mid f_{z_0} = 0 \}$ due to Proposition 2.1. In particular, $\mathcal{H}^v_e$ is regarded as a linear subvariety of $\mathcal{H}_e$, since it is equal to the projectivization of the kernel of the linear map $H^0(\mathbb{P}^N, \mathcal{O}(e)) \to H^0(\mathbb{P}^N, \mathcal{O}(e-1)) : f \mapsto f_{z_0}$.

Let $e^1, \dots, e^r$ be r integers greater than 1. We denote by $\mathcal{H} := \mathcal{H}_{e^1} \times \cdots \times \mathcal{H}_{e^r}$, which parametrizes $(e^1, \dots, e^r)$-complete intersection varieties in $\mathbb{P}^N$. Let us consider

$$\mathcal{H}^v := \mathcal{H}^v_{e^1} \times \cdots \times \mathcal{H}^v_{e^r}, \tag{2}$$

which parametrizes complete intersection varieties being strange for v. Indeed, we have:

$$\mathcal{H}^v = \{ (f^1, \dots, f^r) \in \mathcal{H}) \mid f^1_{z_0} = \cdots = f^r_{z_0} = 0 \}. \tag{3}$$

Note that $\mathcal{H}^v$ is irreducible, since so is each $\mathcal{H}^v_e$.

Remark 3.1. Let $X \subset \mathbb{P}^N$ be an $(e^1, \dots, e^r)$-complete intersection variety being strange for v. Then, from Proposition 2.12, we can find homogeneous polynomials $(f^1, \dots, f^r) \in \mathcal{H}^v$ whose zero set is equal to X.

3.1. **Strange varieties with isolated singular points.** We construct a member of $\mathcal{H}^v$ which defines a complete intersection variety X_0 having an isolated singular point $\alpha \neq v$. Later, the pair (X_0, α) will play an essential role.

For a member $(f^1, \dots, f^r) \in \mathcal{H}^v$ and for $a \in \mathbb{P}^N$, we set

$$D((f^k), a) = D((f^k), a; \mathbb{P}^N) := \begin{bmatrix} f^1_{z_1}(a) & \cdots & f^1_{z_N}(a) \\ \vdots & & \vdots \\ f^r_{z_1}(a) & \cdots & f^r_{z_N}(a) \end{bmatrix}, \tag{4}$$

where we need not consider $f^k_{z_0}$'s since these are zero as in (3). The complete intersection variety $X \subset \mathbb{P}^N$ defined by $(f^1, \dots, f^r)$ is singular at $a \in \mathbb{P}^N$ if and only if $f^1(a) = \cdots = f^r(a) = 0$ and $\operatorname{rk} D((f^k), a) < r$.

Proposition 3.2. *Assume that $\mathcal{H}$ is* not *equal to the parameter space of quadric hypersurfaces in $\mathbb{P}^N$ of odd dimensions in $p = 2$. Assume that $p > 0$ and $e^k \geqslant p$ for some k. Then there exists a complete intersection variety $X_0 \subset \mathbb{P}^N$ defined by a member of $\mathcal{H}^v$ such that $\operatorname{Sing} X_0 \neq \{ v \}$ and $0 < \#(\operatorname{Sing} X_0) < \infty$. In particular, there exists an isolated singular point α of X_0 such that $\alpha \neq v$.*

Remark 3.3 (Hypersurfaces)**.** Let us consider the case where $\mathcal{H}$ is equal to $\mathcal{H}_e$, the parameter space of hypersurfaces in $\mathbb{P}^N$ of degree e.

(a) Suppose that $\mathcal{H} = \mathcal{H}_2$ in $p = 2$, and suppose that N is even (i.e., hypersurfaces are of odd dimensions). Then every member of $\mathcal{H}^v_2$ does *not* satisfy the statement "$\operatorname{Sing} X_0 \neq \{ v \}$ and $0 < \#(\operatorname{Sing} X_0) < \infty$". This is because, if a quadric $X \subset \mathbb{P}^N$ is defined by a member of $\mathcal{H}^v$ and is singular at a point $\alpha \neq v$, then we have $\dim(\operatorname{Sing} X) \geqslant 1$, as follows: Let $\pi : \mathbb{P}^N \setminus \{ \alpha \} \to \mathbb{P}^{N-1}$ be the linear projection from α. Then $Y = \pi(X) \subset \mathbb{P}^{N-1}$ is an $(N-2)$-dimensional quadric, and $X = \overline{\pi^{-1}(Y)}$ (it is a cone with vertex α). Since X is strange for v, Y is strange for $\pi(v)$ as in Remark 2.4(b). From Example 2.2, Y is not smooth. Hence, for $w \in \operatorname{Sing}(Y)$, the line $\overline{\pi^{-1}(w)}$ is contained in $\operatorname{Sing}(X)$.

(b) Suppose that $\mathcal{H} = \mathcal{H}_2$ in $p = 2$, and suppose N in odd. Then $\mathcal{H}^v_2$ satisfies the statement, as follows: We take $L \subset \mathbb{P}^N$ to be a hyperplane containing v, and take Y to be an $(N-2)$-dimensional smooth quadric in L which is strange for v. For a point $\alpha \notin L$, we set $X_0 = \operatorname{Cone}_\alpha(Y)$, the cone over Y with vertex α. Then it follows that $\operatorname{Sing}(X_0) = \{ \alpha \}$ with $\alpha \neq v$.

(c) For $e \geqslant 3$ and $e \geqslant p > 0$, we can construct a member of $\mathcal{H}^v_e$ satisfying the statement of Proposition 3.2, as follows. Suppose that $p \mid e$. Then we set

$X_0 \subset \mathbb{P}^N$ to be the hypersurface defined by

$$f = z_N z_{N-1}^{e-1} + z_{N-1} z_{N-2}^{e-1} + \cdots + z_2 z_1^{e-1} + z_0^e.$$

Then X_0 is strange for v because of $f_{z_0} = 0$. On the other hand, $D(f, z) = \begin{bmatrix} f_{z_1}(z) & \ldots & f_{z_N}(z) \end{bmatrix}$ is equal to

$$\begin{bmatrix} (e-1)z_1^{e-2}z_2 & z_1^{e-1} + (e-1)z_2^{e-2}z_3 & \ldots & z_{N-2}^{e-1} + (e-1)z_N z_{N-1}^{e-2} & z_{N-1}^{e-1} \end{bmatrix}.$$

Then $\operatorname{Sing}(X_0) = \{ (0 : \cdots : 0 : 1) \}$ holds, as follows: We have "$\supset$" immediately. To show "$\subset$", we take $z \in \operatorname{Sing}(X_0)$. Since $D(f, z) = 0$, by the first polynomial from the right of the above description of $D(f, z)$, we have $z_{N-1} = 0$. By the second polynomial from the right, we have $z_{N-2} = 0$ because of $e - 2 \geqslant 1$. Similarly, we have $z_1 = \cdots = z_{N-1} = 0$. Since $f(z) = 0$, we have $z_0 = 0$. Therefore $z = (0 : \cdots : 0 : 1)$.

Suppose that $e > p$ and $p \nmid e$. Then we set $X_0 \subset \mathbb{P}^N$ to be the hypersurface defined by

$$f = z_0^p z_1^{e-p} + z_2^e + \cdots + z_N^e.$$

Then X_0 is strange for v. In addition, $D(f, z)$ is equal to

$$\begin{bmatrix} (e-p)z_0^p z_1^{e-p-1} & e z_2^{e-1} & \ldots & e z_N^{e-1} \end{bmatrix}.$$

Thus $\operatorname{Sing}(X_0) = \{ (1 : 0 : 0 : \cdots : 0), (0 : 1 : 0 : \cdots : 0) \}$ if $e > p + 1$, and $\operatorname{Sing}(X_0) = \{ (0 : 1 : 0 : \cdots : 0) \}$ if $e = p + 1$.

Proof of Proposition 3.2. We have already shown the assertion in the case where $\mathcal{H}$ is the space of hypersurfaces, as above. Thus we assume that $r \geqslant 2$. Without loss of generality, we can assume that $e_1 \geqslant p$. Let us take $Z \subset \mathbb{P}^{N-1}$ to be a complete intersection variety defined by $r-1$ homogeneous polynomials $f^2, \ldots, f^r \in K[z_1, \ldots, z_N]$ of degrees $e^2, \ldots, e^r$, such that $0 < \#(\operatorname{Sing}(Z)) < \infty$ (for example, a cone over a smooth variety). By Bertini's theorem, we can choose a hypersurface $Y^1 \subset \mathbb{P}^{N-1}$ defined by a homogeneous polynomial $f^1 \in K[z_1, \ldots, z_N]$ of degree e^1 such that the intersection $Y := Y^1 \cap Z \subset \mathbb{P}^{N-1}$ is smooth, where $Y^1 \cap \operatorname{Sing}(Z) = \emptyset$.

Let $\beta = (\beta_1 : \cdots : \beta_N) \in \operatorname{Sing}(Z)$. Then we have

$$f^1(\beta) \neq 0 \quad \text{and} \quad \operatorname{rk} D((f^2, \ldots, f^r), \beta; \mathbb{P}^{N-1}) < r - 1.$$

Here, without loss of generality, we can assume that $\beta_1 \neq 0$. We regard Y as a subvariety of $\mathbb{P}^N$ contained in the hyperplane $(z_0 = 0) \simeq \mathbb{P}^{N-1}$.

Now, let us define a complete intersection variety

$$X_0 := (g^1 = f^2 = \cdots = f^r = 0) \subset \mathbb{P}^N,$$

where

$$g^1 := f^1 - \frac{f^1(\beta)}{\beta_1^{e^1-p}} \cdot z_0^p z_1^{e^1-p} \in K[z_0, z_1, \ldots, z_N].$$

Then X_0 is strange for v because of $g^1_{z_0} = 0$. In addition, we have $\dim(\operatorname{Sing}(X_0)) < 1$, as follows: By definition of g^1, the intersection $X_0 \cap (z_0 = 0)$ is equal to the smooth variety Y. It follows that X_0 is smooth at any $x \in Y$ (by considering

regular system of parameters of $\mathcal{O}_{Y,x}$ as in the proof of Proposition 2.12). Thus $\mathrm{Sing}(X_0)$ does not intersect with $(z_0 = 0)$, which implies that $\mathrm{Sing}(X_0)$ must be of dimension < 1.

Next, we set $\alpha := (1 : \beta_1 : \cdots : \beta_N) \in \mathbb{P}^N$, where $\alpha \neq v$. We have $\alpha \in \mathrm{Sing}(X_0)$, as follows: Since $\beta \in Z$, we have $f^2(\alpha) = \cdots = f^r(\alpha) = 0$. By definition of g^1, it follows that $g^1(\alpha) = 0$. Hence $\alpha \in X_0$. Since $\mathrm{rk}\, D((f^2, \dots, f^r), \beta; \mathbb{P}^{N-1}) < r - 1$, we have $\mathrm{rk}\, D((g^1, f^2, \dots, f^r), \alpha; \mathbb{P}^N) < r$. Hence $\alpha \in \mathrm{Sing}(X_0)$. $\square$

3.2. **Irreducible Components of the incidence variety.** Let us consider

$$I = \left\{ ((f^k), a) \in \mathcal{H}^v \times \mathbb{P}^N \;\middle|\; \begin{array}{l} f^1(a) = \cdots = f^r(a) = 0 \\ \text{and } \mathrm{rk}\, D((f^k), a) < r \end{array} \right\},$$

which is the incidence variety parametrizing pairs (X, a) such that X is a complete intersection variety defined by $(f^k) \in \mathcal{H}^v$ and that a is a singular point of X. Let $\mathrm{pr}_1 : I \to \mathcal{H}^v$ and $\mathrm{pr}_2 : I \to \mathbb{P}^N$ be the first and second projections. Here, the image $\mathrm{pr}_1(I) \subset \mathcal{H}^v$ parametrizes singular complete intersection varieties.

For a member $(f^1, \dots, f^r) \in \mathcal{H}^v$, we set

$$D'((f^k), a) := \begin{bmatrix} f^1_{z_1}(a) & \cdots & f^1_{z_{N-1}}(a) \\ \vdots & & \vdots \\ f^r_{z_1}(a) & \cdots & f^r_{z_{N-1}}(a) \end{bmatrix}$$

(i.e., we remove $f^k_{z_N}$'s from $D((f^k), a)$ defined in (4)), and set

$$I' := \left\{ ((f^k), a) \in \mathcal{H}^v \times \mathbb{P}^N \;\middle|\; \begin{array}{l} f^1(a) = \cdots = f^r(a) = 0 \\ \text{and } \mathrm{rk}\, D'((f^k), a) < r \end{array} \right\},$$

where we have $I \subset I'$. We will investigate irreducible components of I' and I.

Lemma 3.4. *Let Λ' be an irreducible component of I'. If $\mathrm{pr}_2(\Lambda') = \mathbb{P}^N$, then Λ' is of dimension $\geqslant \dim(H^v)$.*

Remark 3.5. Let $M_k = M_k(r, n)$ be the set of $(x_{i,j})_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n} \in \mathbb{A}^{rn}$ such that

$$\mathrm{rk} \begin{bmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & & \vdots \\ x_{r,1} & \dots & x_{r,n} \end{bmatrix} \leqslant k.$$

Then M_k is an irreducible subvariety of codimension $(r-k)(n-k)$ in $\mathbb{A}^{rn}$ (see [1, p. 67, II, §2, Prop.]).

Remark 3.6. Let $a = (a_0 : \cdots : a_N) \in \mathbb{P}^N$ be a point with $a_N \neq 0$. Let $\hat{\mathcal{H}}^v_e \subset H^0(\mathbb{P}^N, \mathcal{O}(e))$ be the affine subvariety corresponding to $\mathcal{H}^v_e \subset \mathcal{H}_e$. We consider the morphism

$$\Phi_e : \hat{\mathcal{H}}^v_e \to \mathbb{A}^{N-1} : f \mapsto (f_{z_1}(a), \dots, f_{z_{N-1}}(a)).$$

Then Φ_e is surjective. This is because, for $b = (b_1, \dots, b_{N-1}) \in \mathbb{A}^{N-1}$, we have

$$\Phi_e(b_1/a_N^{e-1} \cdot z_1 z_N^{e-1} + \cdots + b_{N-1}/a_N^{e-1} \cdot z_{N-1} z_N^{e-1}) = b.$$

Proof of Lemma 3.4. Let $a = (a_0 : \cdots : a_N) \in \mathbb{P}^N$ be general. Then we can assume $a_N \neq 0$. Let us calculate the codimension of the intersection $I' \cap (\mathcal{H}^v \times \{ a \})$, which is equal to the fiber of $\mathrm{pr}_2 : I' \to \mathbb{P}^N$ at a. First, we set

$$\Phi = \bigoplus_{1 \leqslant k \leqslant r} \Phi_{e^k} : \hat{\mathcal{H}^v} \to \mathbb{A}^{r(N-1)} : (f^k) \mapsto D'((f^k), a),$$

where $\hat{\mathcal{H}^v} := \bigoplus \hat{\mathcal{H}^v_{e_k}} \subset \bigoplus H^0(\mathbb{P}^N, \mathcal{O}(e^k))$ is the affine subvariety corresponding to $\mathcal{H}^v \subset \mathcal{H}$. From Remark 3.6, Φ is surjective. Moreover, Φ is a smooth morphism, since it is regarded as a linear map of vector spaces. Hence $\Phi^{-1}(M_{r-1}) \subset \hat{\mathcal{H}^v}$ is of codimension $N - r$, where $M_{r-1} = M_{r-1}(r, N-1) \subset \mathbb{A}^{r(N-1)}$ is the subvariety defined in Remark 3.5. Then each irreducible component of

$$\Phi^{-1}(M_{r-1}) \cap F_a \subset \hat{\mathcal{H}^v} \tag{5}$$

is of codimension $\leqslant N$, where $F_a := \{ (f^k) \in \hat{\mathcal{H}^v} \mid f^1(a) = \cdots = f^r(a) = 0 \}$. Now, the projective variety in $\mathcal{H}^v$ corresponding to the affine variety (5) can be identified with

$$I' \cap (\mathcal{H}^v \times \{ a \}) \subset (\mathcal{H}^v \times \{ a \}).$$

Hence each irreducible component of $I' \cap (\mathcal{H}^v \times \{ a \})$ is of codimension $\leqslant N$.

Since $\Lambda' \to \mathbb{P}^N$ is surjective and since $a \in \mathbb{P}^N$ is general, we have that

$$\mathrm{codim}(\Lambda', \mathcal{H}^v \times \mathbb{P}^N) = \mathrm{codim}(\Lambda' \cap (\mathcal{H}^v \times \{ a \}), \mathcal{H}^v \times \{ a \}),$$

and have that $\Lambda' \cap (\mathcal{H}^v \times \{ a \})$ is an irreducible component of $I' \cap (\mathcal{H}^v \times \{ a \})$. Hence $\mathrm{codim}(\Lambda', \mathcal{H}^v \times \mathbb{P}^N) \leqslant N$, which implies the assertion. □

Lemma 3.7. *Let $\Lambda' \subset I'$ be an irreducible subset. If $\mathrm{pr}_2(\Lambda') \not\subset (z_N = 0)$, then $\Lambda' \subset I$.*

Proof. Let $((f^k), a) \in \Lambda'$ be a general member. Since $\mathrm{pr}_2(\Lambda') \not\subset (z_N = 0)$, $a = (a_0 : \cdots : a_N)$ satisfies $a_N \neq 0$. Then, since $f^k_{z_0} = 0$, it follows from Euler's formula $\sum_{j=0}^N a_j f^k_{z_j}(a) = e^k f^k(a) = 0$ that

$$f^k_{z_N}(a) = -(a_1/a_N \cdot f^k_{z_1}(a) + \cdots + a_{N-1}/a_N \cdot f^k_{z_{N-1}}(a))$$

holds for $1 \leqslant k \leqslant r$. Thus the last column vector of the matrix $D((f^k), a)$ defined in (4) is written by a linear combination of the rest of column vectors. Then, since $D'((f^k), a)$ is of rank $< r$, so is $D((f^k), a)$. It follows that $((f^k), a) \in I$. □

Theorem 3.8. *Let $v \in \mathbb{P}^N$ be a point, and let $e^1, \dots, e^r$ be r integers greater than 1. Recall that $\mathcal{H}^v = \mathcal{H}^v_{e^1} \times \cdots \times \mathcal{H}^v_{e^r}$ is the parameter space defined in (2). Assume that $\mathcal{H}$ is* not *equal to the parameter space of quadric hypersurfaces of odd dimensions in $p = 2$. Then $\mathrm{pr}_1(I) = \mathcal{H}^v$, which means that every complete intersection variety defined by a member of $\mathcal{H}^v$ is singular.*

Proof. If $p = 0$ or $e^k < p$ for any k, then it follows from Remark 2.3 and Corollary 2.15 that every X defined by a member of $\mathcal{H}^v$ is a cone with vertex v, in particular, is singular.

Now we assume that $p > 0$ and $e^k \geqslant p$ for some k. From Proposition 3.2, we can take $((f^k), \alpha) \in \mathcal{H}^v \times \mathbb{P}^N$ such that $\alpha \neq v$ is an isolated singular point of the complete intersection variety $X_0 \subset \mathbb{P}^N$ defined by (f^k).

We denote by $\mathrm{PGL}(\mathbb{P}^N; v) \subset \mathrm{PGL}(\mathbb{P}^N)$ the group of automorphisms σ of $\mathbb{P}^N$ such that $\sigma(v) = v$. Let us consider the subset of I parametrizing pairs $(\sigma(X_0), \sigma(\alpha))$ with $\sigma \in \mathrm{PGL}(\mathbb{P}^N; v)$, which is actually given by the orbit of $((f^k), \alpha)$ in I,

$$\{ (((\sigma^{-1})^* f^k), \sigma(\alpha)) \in I \mid \sigma \in \mathrm{PGL}(\mathbb{P}^N; v) \},$$

where $(\sigma^{-1})^* f^k(z) := f^k(\sigma^{-1}(z))$. Now we take an irreducible component Λ of I containing the above orbit. Then $\mathrm{pr}_2(\Lambda) = \mathbb{P}^N$. Here we have

$$\dim(\Lambda) = \dim(\mathrm{pr}_1(\Lambda)). \tag{6}$$

The reason is as follows: The fiber of $\mathrm{pr}_1 : I \to \mathcal{H}^v$ at (f^k) is equal to $\mathrm{Sing}(X_0)$. Since α is isolated, the set $\{ \alpha \}$ is an irreducible component of $\mathrm{Sing}(X_0)$. Since each irreducible component of a fiber of $\Lambda \to \mathrm{pr}_1(\Lambda)$ must be of dimension $\geqslant \dim(\Lambda) - \dim(\mathrm{pr}_1(\Lambda))$, the equality $\dim(\Lambda) = \dim(\mathrm{pr}_1(\Lambda))$ holds.

Let $\Lambda' \subset I'$ be an irreducible component of I' such that $\Lambda \subset \Lambda'$. From Lemma 3.4, $\dim(\Lambda') \geqslant \dim(\mathcal{H}^v)$. From Lemma 3.7, we have $\Lambda = \Lambda'$. From (6), we have $\dim(\mathrm{pr}_1(\Lambda)) \geqslant \dim(\mathcal{H}^v)$, and hence $\mathrm{pr}_1(\Lambda) = \mathcal{H}^v$. $\square$

Proof of Theorem 1.1. The equivalence "(a) $\Leftrightarrow$ (c)" follows immediately from Proposition 2.9. The implication "(b) $\Rightarrow$ (a)" follows as in Example 2.2. Now we show the implication "(a) $\Rightarrow$ (b)". Let $X \subset \mathbb{P}^N$ be a smooth $(e^1, \dots, e^r)$-complete intersection variety, and assume that X is strange for a point $v \in \mathbb{P}^N$. It is sufficient to consider the case where X is non-degenerate. Then $e^k > 1$ for any k. As in Remark 3.1, it follows from Proposition 2.12 that X is defined by a member of $(f^k) \in \mathcal{H}^v$. Since X is smooth, it follows from Theorem 3.8 that X must be a quadric of odd dimension in $p = 2$. $\square$

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References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, Geometry of Algebraic Curves, Volume I. Springer 1985.

[2] A. van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, vol. **190**, Birkhauser Verlag, Basel, 2000.

[3] A. Holme and E. Lluis, Strange varieties, Appendix to "Embeddings, projective invariants and classifications", Monografias del Instituto de Matematicas, Univ. Nac. Autonoma de Mexico. Vol. 7, 1979, pp. 93–122.

[4] S. L. Kleiman and R. Piene, On the inseparability of the Gauss map, "Enumerative Algebraic Geometry (Proceedings of the 1989 Zeuthen Symposium)", Contemp. Math. **123**, Amer. Math. Soc., Providence, 1991, pp. 107–129.

[5] E. Lluis: Variedades algebraicas con ciertas condiciones en sus tangentes, Bol. Soc. Mat. Mexicana (2) **7** (1962), 47–56.

[6] H. Matsumura, Commutative ring theory, Cambridge Stud. Adv. Math., vol. **8**, Cambridge University Press, Cambridge, 1989.

[7] S. Mori, Projective manifolds with ample tangent bundles, Ann. Math. (2) **110**(3) (1979), 593–606.

[8] S. Okuda, Kernels of derivations in positive characteristic, Hiroshima Math. J. **34** (2004), 1–19

[9] P. Samuel: Lectures on old and new results on algebraic curves, Tata Inst. Fund. Res. Lectures on Math. **36**, Tata Inst. Fund. Res., Bombay 1966.

[10] J. M. Wahl, A cohomological characterization of P^n, Invent. Math, **72** (1983), 315–322.

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